26

INTEGRATION

In the previous lesson, you have learnt the concept of derivative of a function. You have also learnt the application of derivative in various situations.

Consider the reverse problem of finding the original function, when its derivative (in the form of a function) is given. This reverse process is given the name of integration. In this lesson, we shall study this concept and various methods and techniques of integration.

OBJECTIVES

After studying this lesson, you will be able to :

- explain integration as inverse process (anti-derivative) of differentiation;
- find the integral of simple functions like $x^n, \sin x, \cos x,$

  $\sec^2 x, \operatorname{cosec}^2 x, \sec x \tan x, \operatorname{cosec} x \cot x, \frac{1}{x}, e^x$ etc.;
- state the following results :

  (i) $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

  (ii) $\int [\pm k f(x)]\, dx = \pm k \int f(x)\, dx$
- find the integrals of algebraic, trigonometric, inverse trigonometric and exponential functions;
- find the integrals of functions by substitution method.
- evaluate integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{a^2 - x^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c},$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{(px+q)\,dx}{ax^2 + bx + c}, \int \frac{(px+q)\,dx}{\sqrt{ax^2 + bx + c}}$$

- derive and use the result

$$\int \frac{f'(x)}{f(x)} = \ell n |f(x)| + C$$

- state and use the method of integration by parts;
- evaluate integrals of the type :

$$\int \sqrt{x^2 \pm a^2}dx, \int \sqrt{a^2 - x^2}dx, \int e^{ax} \sin bx\, dx, \int e^{ax} \cos bx\, dx,$$

$$\int (px+q)\sqrt{ax^2+bx+c}\, dx, \int \sin^{-1} x\, dx, \int \cos^{-1} x\, dx,$$

$$\int \sin^n x \cos^m x\, dx, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}$$

- derive and use the result

$$\int e^x [f(x) + f'(x)]dx = e^x f(x) + c \text{; and}$$

- integrate rational expressions using partial fractions.

EXPECTED BACKGROUND KNOWLEDGE

- Differentiation of various functions
- Basic knowledge of plane geometry
- Factorization of algebraic expression
- Knowledge of inverse trigonometric functions

26.1 INTEGRATION

Integration literally means summation. Consider, the problem of finding area of region ALMB as shown in Fig. 26.1.

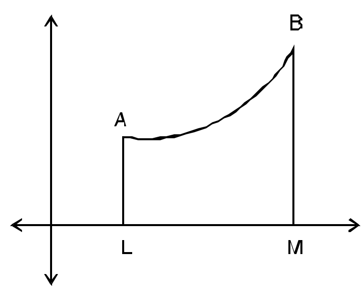


Fig. 26.1

We will try to find this area by some practical method. But that may not help every time. To solve such a problem, we take the help of integration (summation) of area. For that, we divide the figure into small rectangles (See Fig.26.2).

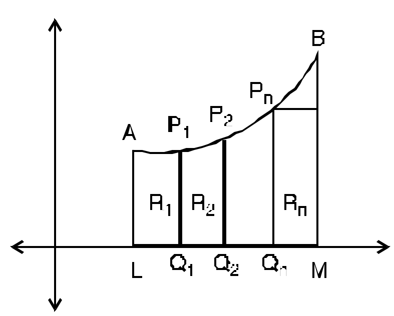


Fig. 26.2

Unless these rectangles are having their width smaller than the smallest possible, we cannot find the area.

This is the technique which Archimedes used two thousand years ago for finding areas, volumes, etc. The names of Newton (1642-1727) and Leibnitz (1646-1716) are often mentioned as the creators of present day of Calculus.

The integral calculus is the study of integration of functions. This finds extensive applications in Geometry, Mechanics, Natural sciences and other disciplines.

In this lesson, we shall learn about methods of integrating polynomial, trigonometric, exponential and logarithmic and rational functions using different techniques of integration.

26.2. INTEGRATION AS INVERSE OF DIFFERENTIATION

Consider the following examples :

(i) $\frac{d}{dx}\left(x^2\right) = 2x$ (ii) $\frac{d}{dx}(\sin x) = \cos x$ (iii) $\frac{d}{dx}\left(e^x\right) = e^x$

Let us consider the above examples in a different perspective

(i) 2x is a function obtained by differentiation of x^2.

$\Rightarrow x^2$ is called the antiderivative of $2x$

(ii) cos x is a function obtained by differentiation of sin x

$\Rightarrow$ sin x is called the antiderivative of cos x

(iii) Similarly, e^x is called the antiderivative of e^x

Generally we express the notion of antiderivative in terms of an operation. This operation is called the operation of integration. We write

1. Integration of 2 x is x^2
2. Integration of cos x is sin x
3. Integration of e^x is e^x

The operation of integration is denoted by the symbol $\int$.

Thus

1. $\int 2x\,dx = x^2$ 2. $\int \cos x\,dx = \sin x$ 3. $\int e^x\,dx = e^x$

Remember that dx is symbol which together with symbol $\int$ denotes the operation of integration.

The function to be integrated is enclosed between $\int$ and dx.

Definition : If $\frac{d}{dx}[f(x)] = f'(x)$, then f (x) is said to be an integral of $f'(x)$ and is written as $\int f'(x)dx = f(x)$

The function f '(x) which is integrated is called the integrand.

Constant of integration

$$\text{If } y = x^2, \text{then } \frac{dy}{dx} = 2x$$

$$\therefore \quad \int 2x\,dx = x^2$$

Now consider $\frac{d}{dx}\left(x^2 + 2\right)$ or $\frac{d}{dx}\left(x^2 + c\right)$ where c is any real constant . Thus, we see that integral of 2x is not unique. The different values of $\int 2x\,dx$ differ by some constant. Therefore, $\int 2x\,dx = x^2 + C$, where c is called the constant of integration.

Thus $\int e^x\,dx = e^x + C$, $\int \cos x\,dx = \sin x + c$

In general $\int f'(x)dx = f(x) + C$. The constant c can take any value.

We observe that the derivative of an integral is equal to the integrand.

Note : $\int f(x)dx$, $\int f(y)dy$, $\int f(z)dz$ but not like $\int f(z)dx$

Example 26.1 Find the integral of the following :

(i) x^3 (ii) x^{30} (iii) x^n

Solution :

(i) $\int x^3 dx = \frac{x^4}{4} + C$, since $\frac{d}{dx}\left(\frac{x^4}{4}\right) = \frac{4x^3}{4} = x^3$

(ii) $\int x^{30} dx = \frac{x^{31}}{31} + C$, since $\frac{d}{dx}\left(\frac{x^{31}}{31}\right) = \frac{31x^{30}}{31} = x^{30}$

(iii) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

since $\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{1}{n+1}\frac{d}{dx}x^{n+1} = \frac{1}{n+1}(n+1)x^n = x^n$

Example 26.2 (i) If $\frac{dy}{dx} = \cos x$, find y. (ii) If $\frac{dy}{dx} = \sin x$, find y.

Solution : (i) $\int \frac{dy}{dx} dx = \int \cos x\,dx \quad \Rightarrow \quad y = \sin x + C$

(ii) $\int \frac{dy}{dx} dx = \int \sin x\,dx \quad \Rightarrow \quad y = -\cos x + C$

26.3 INTEGRATION OF SIMPLE FUNCTIONS

We have already seen that if f (x) is any integral of f '(x), then functions of the form f (x) + C provide integral of f '(x). We repeat that C can take any value including 0 and thus

$$\int f'(x)dx = f(x) + C$$

which is an indefinite integral and it becomes a definite integral with a defined value of C.

Example 26.3 Write any 4 different values of $\int 4x^3 dx$.

Solution : $\int 4x^3 dx = x^4 + C$, where C is a constant.

The four different values of $\int 4x^3 dx$ may be $x^4 + 1, x^4 + 2, x^4 + 3$ and $x^4 + 4$ etc.

Integrals of some simple functions given below. The validity of the integrals is checked by showing that the derivative of the integral is equal to the integrand.

	Integral	Verification
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where n is a constant and $n \neq -1$.	$\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$
2.	$\int \sin x\, dx = -\cos x + C$	$\because \frac{d}{dx}(-\cos x + C) = \sin x$
3.	$\int \cos x\, dx = \sin x + C$	$\because \frac{d}{dx}(\sin x + C) = \cos x$
4.	$\int \sec^2 x\, dx = \tan x + C$	$\because \frac{d}{dx}(\tan x + C) = \sec^2 x$
5.	$\int \operatorname{cosec}^2 x\, dx = -\cot x + C$	$\because \frac{d}{dx}(-\cot x + C) = \operatorname{cosec}^2 x$
6.	$\int \sec x \tan x\, dx = \sec x + C$	$\because \frac{d}{dx}(\sec x + C) = \sec x \tan x$
7.	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$	$\because \frac{d}{dx}(-\operatorname{cosec} x + C) = \operatorname{cosec} x \cot x$
8.	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$	$\because \frac{d}{dx}(\sin^{-1} x + C) = \frac{1}{\sqrt{1-x^2}}$
9.	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$	$\because \frac{d}{dx}(\tan^{-1} x + C) = \frac{1}{1+x^2}$
10.	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$	$\because \frac{d}{dx}(\sec^{-1} x + C) = \frac{1}{x\sqrt{x^2-1}}$

11. $\int e^x dx = e^x + C$ $\quad \because \frac{d}{dx}\left(e^x + C\right) = e^x$

12. $\int a^x dx = \frac{a^x}{\log a} + C$ $\quad \because \frac{d}{dx}\left(\frac{a^x}{\log a} + C\right) = a^x = \frac{1}{x}$ if $x > 0$

13. $\int \frac{1}{x} dx = \log|x| + C$ $\quad \because \frac{d}{dx}\left(\log|x| + C\right)$

WORKING RULE

1. To find the integral of x^n, increase the index of x by 1, divide the result by new index and add constant C to it.

2. $\int \frac{1}{f(x)} dx$ will be very often written as $\int \frac{dx}{f(x)}$.

CHECK YOUR PROGRESS 26.1

1. Write any five different values of $\int x^{\frac{5}{2}} dx$

2. Write indefinite integral of the following :

   (a) x^5 (b) $\cos x$ (c) 0

3. Evaluate :

   (a) $\int x^6 dx$ (b) $\int x^{-7} dx$ (c) $\int \frac{1}{x} dx$ (d) $\int 3^x 5^{-x} dx$

   (e) $\int \sqrt[3]{x}\, dx$ (f) $\int x^{-9}\, dx$ (g) $\int \frac{1}{\sqrt{x}} dx$ (h) $\int \sqrt[9]{x^{-8}}\, dx$

4. Evaluate :

   (a) $\int \frac{\cos\theta}{\sin^2\theta} d\theta$ (b) $\int \frac{\sin\theta}{\cos^2\theta} d\theta$

   (c) $\int \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} d\theta$ (d) $\int \frac{1}{\sin^2\theta} d\theta$

26.4 PROPERTIES OF INTEGRALS

If a function can be expressed as a sum of two or more functions then we can write the integral of such a function as the sum of the integral of the component functions, e.g. if $f(x) = x^7 + x^3$, then

$$\int f(x)\,dx = \int [x^7 + x^3]\,dx$$

$$= \int x^7 dx + \int x^3 dx$$

$$= \frac{x^8}{8} + \frac{x^4}{4} + C$$

So, in general the integral of the sum of two functions is equal to the sum of their integrals.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Similarly, if the given function

$$f(x) = x^7 - x^2$$

we can write it as $\int f(x)\,dx = \int \left(x^7 - x^2\right) dx$

$$= \int x^7 dx - \int x^2 dx$$

$$= \frac{x^8}{8} - \frac{x^3}{3} + C$$

The integral of the difference of two functions is equal to the difference of their integrals.

i.e. $$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

If we have a function f(x) as a product of a constant (k) and another function $[g(x)]$

i.e. $f(x) = kg(x)$, then we can integrate f(x) as

$$\int f(x)\,dx = \int kg(x)\,dx$$

$$= k\int g(x)\,dx$$

Integral of product of a constant and a function is product of that constant and integral of the function.

i.e. $$\int kf(x)\,dx = k\int f(x)\,dx$$

Example 26.4 Evaluate :

(ii) $\int 4^x\,dx$ (ii) $\int (2^x)(3^{-x})\,dx$

Solution :(i) $$\int 4^x\,dx = \frac{4^x}{\log 4} + C$$

(ii) $$\int (2^x)(3^{-x})\,dx = \int \frac{2^x}{3^x}\,dx$$

$$= \int \left(\frac{2}{3}\right)^x dx = \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Remember in (ii) it would not be correct to say that

$$\int 2^x 3^{-x} dx = \int 2^x dx \int 3^{-x} dx$$

Because $\quad \int 2^x dx \int 3^{-x} dx = \frac{2^x}{\log 2}\left(\frac{3^{-x}}{\log 3}\right) + C \neq \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$

Therefore, **integral of a product of two functions is not always equal to the product of the integrals.** We shall deal with the integral of a product in a subsequent lesson.

Example 26.5 Evaluate :

(i) $\int \frac{dx}{\cos^n x}$, when n = 0 and n = 2 (ii) $\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} d\theta$

Solution :

(i) When n = 0, $\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^o x}$

$$= \int \frac{dx}{1} = \int dx$$

Now $\int dx$ can be written as $\int x^o dx$.

$\therefore \quad \int dx = \int x^o dx$

$$= \frac{x^{o+1}}{0+1} + C = x + C$$

When n = 2,

$$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^2 x}$$

$$= \int \sec^2 x \, dx$$

$$= \tan x = C$$

(ii) $\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} d\theta = \int \frac{-1}{\sin^2\theta} d\theta$

$$= -\int \operatorname{cosec}^2\theta \, d\theta$$

$$= \cot\theta + C$$

Example 26.6 Evaluate :

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \frac{x^2+1}{x^3}\, dx$

(iii) $\int \frac{1-x}{\sqrt{x}}\, dx$ (iv) $\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx$

Solution : (i) $\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx = -\cos x + \sin x + C$

(ii) $$\int \frac{x^2+1}{x^3}\, dx = \int \left(\frac{x^2}{x^3} + \frac{1}{x^3} \right) dx$$

$$= \int \frac{1}{x}\, dx + \int \frac{1}{x^3}\, dx$$

$$= \log|x| + \frac{x^{-3+1}}{-3+1} + C$$

$$= \log|x| - \frac{1}{2x^2} + C$$

(iii) $$\int \frac{1-x}{\sqrt{x}}\, dx = \int \left(\frac{1}{\sqrt{x}} - \frac{x}{\sqrt{x}} \right) dx$$

$$= \int \left(x^{-\frac{1}{2}} - x^{\frac{1}{2}} \right) dx$$

$$= 2\sqrt{x} - \frac{2}{3} x^{3/2} + C$$

(iv) $$\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx = \int \frac{dx}{1+x^2} - \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \tan^{-1} x - \sin^{-1} x + C$$

Example 26.7 Evaluate :

(i) $\int \sqrt{1-\sin 2\theta}\, d\theta$ (ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}} \right) dx$

(iii) $\int (\tan x + \cot x)^2\, dx$ (iv) $\int \left(\frac{x^6-1}{x^2-1} \right) dx$

Solution : (i) $\sqrt{1-\sin 2\theta} = \sqrt{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}$

$\left[\because \sin^2\theta + \cos^2\theta = 1 \right]$

$$= \sqrt{(\cos\theta - \sin\theta)^2}$$

$$= \pm(\cos\theta - \sin\theta)$$

(sign is selected depending upon the value of θ)

(a) If $\sqrt{1-\sin 2\theta} = \cos\theta - \sin\theta$

then $\int \sqrt{1-\sin 2\theta}\, d\theta = \int (\cos\theta - \sin\theta)\, d\theta$

$$= \int \cos\theta\, d\theta - \int \sin\theta\, d\theta$$

$$= \sin\theta + \cos\theta + C$$

(b) If $\int \sqrt{1-\sin 2\theta}\, d\theta = \int (-\cos\theta + \sin\theta)\, d\theta$

$$= -\int \cos\theta\, d\theta + \int \sin\theta\, d\theta$$

$$= -\sin\theta - \cos\theta + C$$

(ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}}\right) dx = \int 4e^x dx - \int \frac{3}{x\sqrt{x^2-1}} dx$

$$= 4\int e^x dx - 3\int \frac{dx}{x\sqrt{x^2-1}}$$

$$= 4e^x - 3\sec^{-1} x + C$$

(iii) $\int (\tan x + \cot x)^2 dx = \int \left(\tan^2 x + \cot^2 x + 2\tan x \cot x\right) dx$

$$= \int \left(\tan^2 x + \cot^2 x + 2\right) dx$$

$$= \int \left(\tan^2 x + 1 + \cot^2 x + 1\right) dx$$

$$= \int \left(\sec^2 x + \operatorname{cosec}^2 x\right) dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x + C$$

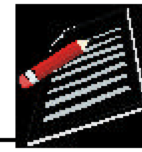
(iv) $\int \left(\frac{x^6 - 1}{x^2 + 1}\right) dx = \int \left(x^4 - x^2 + 1 - \frac{2}{x^2+1}\right) dx$ (dividing $x^6 - 1$ by $x^2 + 1$)

$$= \int x^4 dx - \int x^2 dx + \int dx - 2\int \frac{dx}{x^2+1}$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1} x + C$$

Example 26.8 Evaluate :

(i) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 dx$ (ii) $\int \left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}}\right) dx$ (iii) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}$



Solution :

(i) $$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3 dx=\int\left(x^{3/2}+3x\frac{1}{\sqrt{x}}+3\sqrt{x}\,\frac{1}{x}+\frac{1}{x^{3/2}}\right)dx$$

$$=\int x^{3/2}dx+3\int\sqrt{x}\,dx+3\int\frac{1}{\sqrt{x}}dx+\int\frac{dx}{x^{3/2}}$$

$$=\frac{x^{5/2}}{\frac{5}{2}}+3\frac{x^{3/2}}{\frac{3}{2}}+3\frac{x^{1/2}}{\frac{1}{2}}-\frac{2}{\sqrt{x}}+C$$

$$=\frac{2}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+6x^{\frac{1}{2}}-2x^{-\frac{1}{2}}+C$$

(ii) $$\int\left(\frac{4e^{5x}-9e^{4x}-3}{e^{3x}}\right)dx=\int\frac{4e^{5x}}{e^{3x}}dx-\int\frac{9e^{4x}}{e^{3x}}dx-\int\frac{3dx}{e^{3x}}$$

$$=4\int e^{2x}\,dx-9\int e^{x}\,dx-3\int e^{-3x}\,dx$$

$$=2e^{2x}-9e^{x}+e^{-3x}+C$$

(iii) $$\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$

Let us first rationalise the denominator of the integrand.

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}$$

$\therefore$ $$\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}}=\int\frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}dx$$

$$=\int\frac{\sqrt{x+a}}{a-b}dx-\int\frac{\sqrt{x+b}}{a-b}dx$$

$$=\frac{1}{a-b}\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{1}{a-b}\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}+C$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

Notes

CHECK YOUR PROGRESS 26.2

1. Evaluate :

(a) $\int \left(x + \frac{1}{2} \right) dx$ (b) $\int \frac{-x^2}{1+x^2} dx$

(c) $\int \left(10x^9 - \sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$ (d) $\int \left(\frac{5+3x-6x^2-7x^4-8x^6}{x^6} \right) dx$

(e) $\int \frac{x^4}{1+x^2} dx$ (f) $\int \left(\sqrt{x} + \frac{2}{\sqrt{x}} \right)^2 dx$

2. Evaluate :

(a) $\int \frac{dx}{1+\cos 2x}$ (b) $\int \tan^2 x\,dx$ (c) $\int \frac{2\cos x}{\sin^2 x} dx$

(d) $\int \frac{dx}{1-\cos 2x}$ (e) $\int \frac{\sin x}{\cos^2 x} dx$ (f) $\int (\operatorname{cosec} x - \cot x) \operatorname{cosec} x\,dx$

3. Evaluate :

(a) $\int \sqrt{1+\cos 2x}\,dx$ (b) $\int \sqrt{1-\cos 2x}\,dx$ (c) $\int \frac{1}{1-\cos 2x} dx$

4. Evaluate :

(a) $\int \sqrt{x+2}\,dx$ (b) $\int \frac{17}{(x+1)^{17}} dx$

We know that

$$\frac{d}{dx}(ax+b)^{n+1} = (n+1)a(ax+b)^n$$

$$\Rightarrow \quad (ax+b)^n = \frac{1}{(n+1)a} \frac{d}{dx}(ax+b)^{n+1}, \; n \neq -1$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C$$

To find the integral of $(ax+b)^n$, increase the index by one and divide the result by the increased index and the coefficient of x and add a constant of integration.

Example 26.9 Integrate the following :

(i) $(x+9)^{11}$ (ii) e^{x+7} (iii) $\cos(x+5\pi)$

(iv) $\operatorname{cosec}^2(2x+3)$

Solution :

(i) $$\int (x+9)^{11}\,dx = \frac{(x+9)^{11+1}}{11+1} + C$$

$$= \frac{(x+9)^{12}}{12} + C$$

(ii) $$\int e^{x+7}dx = \int e^{x}\cdot e^{7}\,dx$$

$$= e^{7}\int e^{x}\,dx \qquad (e^{7} \text{ is a constant quantity})$$

$$= e^{7}\cdot e^{x} + C = e^{x+7} + C$$

(iii) $$\int \cos(x+5\pi)\,dx = \sin(x+5\pi) + C$$

$$\left[\because \frac{d}{dx}\left(\sin(x+5\pi)+C\right) = \cos(x+5\pi)\frac{d}{dx}(x+5\pi) = \cos(x+5\pi)\right]$$

(iv) $$\int \text{cosec}^2(2x+3)\,dx = \frac{-\cot(2x+3)}{2} + C$$

$$\left[\because \frac{d}{dx}\left(\frac{-\cot(2x+3)}{2}+C\right) = 2\left(\frac{-(-\text{cosec}^2(2x+3))}{2}\right) = \text{cosec}^2(2x+3)\right]$$

Example 26.10 Evaluate : $\int (2x+5)(x-1)\,x^{\frac{-2}{3}}\,dx$

Solution : $$\int (2x+5)(x-1)\,x^{\frac{-2}{3}}\,dx = \int\left(2x^2+3x-5\right)x^{\frac{-2}{3}}\,dx$$

$$= \int\left(2x^{2-\frac{2}{3}} + 3x^{1-\frac{2}{3}} - 5x^{\frac{-2}{3}}\right)dx$$

$$= 2\int x^{4/3}dx + 3\int x^{1/3}dx - 5\int x^{-2/3}dx$$

$$= 2\frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + 3\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 5\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C$$

$$= \frac{6}{7}x^{\frac{7}{3}} + \frac{9}{4}x^{\frac{4}{3}} - 15x^{\frac{1}{3}} + C$$

Example 26.11 Evaluate : $\int \frac{x^2e^x + (x+1)^2}{x^2}\,dx$

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Solution : $$\int \frac{x^2 e^x + (x+1)^2}{x^2}\,dx = \int \frac{x^2 e^x + x^2 + 2x + 1}{x^2}\,dx$$

$$= \int \left(e^x + 1 + \frac{2}{x} + \frac{1}{x^2} \right) dx$$

$$= \int e^x dx + \int dx + 2\int \frac{dx}{x} + \int \frac{dx}{x^2}$$

$$= e^x + x + 2\log|x| - \frac{1}{x} + C$$

Example 26.12 Integrate $\sec^2 x \operatorname{cosec}^2 x$ w.r.t. x

Solution : $$\int \sec^2 x \operatorname{cosec}^2 x\, dx = \int (1+\tan^2 x)(1+\cot^2 x)\, dx$$

$$\left[\because \sec^2 x = 1+\tan^2 x \text{ and } \operatorname{cosec}^2 x = 1+\cot^2 x\right]$$

$$= \int (1+\tan^2 x + \cot^2 x + \tan^2 x \cot^2 x)\, dx$$

$$= \int (1+\tan^2 x) + (1+\cot^2 x)\, dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$$

$$= \tan x - \cot x + C$$

Example 26.13 Evaluate : $\int \left(\frac{-7}{1+x^2} - \frac{6}{\sqrt{1-x^2}} \right) dx$

Solution : $$\int \left(\frac{-7}{1+x^2} - \frac{6}{\sqrt{1-x^2}} \right) dx = -7\int \frac{dx}{1+x^2} - 6\int \frac{dx}{\sqrt{1-x^2}}$$

$$= -7\tan^{-1} x - 6\sin^{-1} x + C$$

Example 26.14 Evaluate : $\int (x + \cos x)\, dx$

Solution : $$\int (x+\cos x)\, dx = \int x\, dx + \int \cos x\, dx$$

$$= \frac{x^2}{2} + \sin x + C$$

26.5 TECHNIQUES OF INTEGRATION

26.5.1 Integration By Substitution

This method consists of expressing $\int f(x)\, dx$ in terms of another variable so that the resultant function can be integrated using one of the standard results discussed in the previous lesson. First, we will consider the functions of the type $f(ax+b),\ a \neq 0$ where f (x) is a standard function.

Example 26.15 Evaluate :

(i) $\int \sin(ax+b)\,dx$ (ii) $\int \cos\left(7x+\frac{\pi}{4}\right)dx$ (iii) $\int \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

Solution : (i) $\int \sin(ax+b)\,dx$

Put $ax+b=t.$

Then $a=\frac{dt}{dx}$ or $dx=\frac{dt}{a}$

$\therefore$ $\int \sin(ax+b)\,dx = \int \sin t\,\frac{dt}{a}$ (Here the integration factor will be replaced by dt.)

$$=\frac{1}{a}\int \sin t\,dt$$

$$=\frac{1}{a}(-\cos t)+C$$

$$=-\frac{\cos(ax+b)}{a}+C$$

(ii) $\int \cos\left(7x+\frac{\pi}{4}\right)dx$

Put $7x+\frac{\pi}{4}=t \Rightarrow 7dx=dt$

$\therefore$ $\int \cos\left(7x+\frac{\pi}{4}\right)dx=\int \cos t\,\frac{dt}{7}$

$$=\frac{1}{7}\int \cos t\,dt$$

$$=\frac{1}{7}\sin t+C$$

$$=\frac{1}{7}\sin\left(7x+\frac{\pi}{4}\right)+C$$

(iii) $\int \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

Put $\frac{\pi}{4}-\frac{x}{2}=t$

Then $\quad -\frac{1}{2} = \frac{dt}{dx} \quad$ or $\quad dx = -2dt$

$$\therefore \quad \int \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) dx = -2\int \sin t \, dt$$

$$= -2(-\cos t) + C$$

$$= 2\cos t + C$$

$$= 2\cos\left(\frac{\pi}{4} - \frac{x}{2}\right) + C$$

Similarly, the integrals of the following functions will be—

$$\int \sin 2x \, dx = -\frac{1}{2}\cos 2x + C$$

$$\int \sin\left(3x + \frac{\pi}{3}\right) dx = -\frac{1}{3}\cos\left(3x + \frac{\pi}{3}\right) + C$$

$$\int \sin\left(\frac{\pi}{4} - \frac{x}{4}\right) dx = 4\cos\left(\frac{\pi}{4} - \frac{x}{4}\right) + C$$

$$\int \cos(ax + b) dx = \frac{1}{a}\sin(ax + b) + C$$

$$\int \cos 2x \, dx = \frac{1}{2}\sin 2x + C$$

Example 26.16 Evaluate :

(i) $\int (ax + b)^n dx$, where $n \neq -1$ (ii) $\int \frac{1}{(ax + b)} dx$

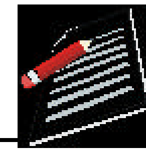
Solution : (i) $\int (ax + b)^n dx$, where $n \neq -1$

Put $\quad ax + b = t \quad \Rightarrow \quad a = \frac{dt}{dx}$ or $dx = \frac{dt}{a}$

$$\therefore \quad \int (ax + b)^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a} \cdot \frac{t^{n+1}}{(n+1)} + C$$

$$= \frac{1}{a} \cdot \frac{(ax + b)^{n+1}}{n+1} + C \quad \text{where } n \neq -1$$



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(ii) $\int \frac{1}{(ax+b)} dx$

Put $\quad ax + b = t \quad \Rightarrow \quad dx = \frac{1}{a} dt$

$$\therefore \quad \int \frac{1}{(ax+b)} dx = \int \frac{1}{a} \cdot \frac{dt}{t}$$

$$= \frac{1}{a} \log |t| + C$$

$$= \frac{1}{a} \log |ax + b| + C$$

Example 26.17 Evaluate :

(i) $\int e^{5x+7} dx$ $\qquad$ (ii) $\int e^{-3x-3} dx$

Solution : (i) $\int e^{5x+7} dx$

Put $\quad 5x + 7 = t \quad \Rightarrow \quad dx = \frac{dt}{5}$

$$\therefore \quad \int e^{5x+7} dx = \frac{1}{5} \int e^t dt$$

$$= \frac{1}{5} e^t + C$$

$$= \frac{1}{5} e^{5x+7} + C$$

(ii) $\int e^{-3x-3} dx$

Put $\quad -3x - 3 = t \quad \Rightarrow \quad dx = \frac{1}{-3} dt$

$$\therefore \quad \int e^{-3x-3} dx = -\frac{1}{3} \int e^t dt$$

$$= -\frac{1}{3} e^t + C$$

$$= -\frac{1}{3} e^{-3x-3} + C$$

Likewise $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

Similarly, using the substitution ax + b = t, the integrals of the following functions will be :

$$\int (ax+b)^n \, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{(ax+b)} dx = \frac{1}{a} \log |ax+b| + C$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$$

$$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$$

$$\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b) \, dx = -\frac{1}{a} \cot(ax+b) + C$$

$$\int \sec(ax+b) \tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b) \cot(ax+b) \, dx = -\frac{1}{a} \operatorname{cosec}(ax+b) + C$$

Example 26.18 Evaluate :

(i) $\int \sin^2 x \, dx$ (ii) $\int \sin^3 x \, dx$ (iii) $\int \cos^3 x \, dx$ (iv) $\int \sin 3x \sin 2x \, dx$

Solution : We use trigonometrical identities and express the functions in terms of sines and cosines of multiples of x

(i) $$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int 1 dx - \frac{1}{2} \int \cos 2x \, dx$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

(ii) $$\int \sin^3 x \, dx = \int \frac{3\sin x - \sin 3x}{4} dx \qquad \left[\because \sin 3x = 3\sin x - 4\sin^3 x\right]$$

$$= \frac{1}{4} \int (3\sin x - \sin 3x) \, dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$$

(iii) $$\int \cos^3 x \, dx = \int \frac{\cos 3x + 3\cos x}{4} dx \quad [\because \cos 3x = 4\cos^3 x - 3\cos x]$$

$$= \frac{1}{4}\int (\cos 3x + 3\cos x)\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] + C$$

(iv) $$\int \sin 3x \sin 2x \, dx = \frac{1}{2}\int 2\sin 3x \sin 2x \, dx$$

$$[\because 2\sin A \sin B = \cos(A - B) - \cos(A + B)]$$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + C$$

CHECK YOUR PROGRESS 26.3

1. Evaluate :

   (a) $\int \sin(4 - 5x)\, dx$ (b) $\int \sec^2(2 + 3x)\, dx$

   (c) $\int \sec\left(x + \frac{\pi}{4}\right) dx$ (d) $\int \cos(4x + 5)\, dx$

   (e) $\int \sec(3x + 5)\tan(3x + 5)\, dx$

   (f) $\int \operatorname{cosec}(2 + 5x)\cot(2 + 5x)\, dx$

2. Evaluate :

   (a) $\int \frac{dx}{(3 - 4x)^4}$ (b) $\int (x + 1)^4\, dx$ (c) $\int (4 - 7x)^{10}\, dx$

   (d) $\int (4x - 5)^3\, dx$ (e) $\int \frac{1}{3x - 5}\, dx$ (f) $\int \frac{1}{\sqrt{5 - 9x}}\, dx$

   (g) $\int (2x + 1)^2\, dx$ (h) $\int \frac{1}{x + 1}\, dx$

3. Evaluate :

   (a) $\int e^{2x+1} dx$ (b) $\int e^{3-8x} dx$ (c) $\int \frac{1}{e^{(7+4x)}}\, dx$

4. Evaluate :

(a) $\int \cos^2 x dx$ (b) $\int \sin^3 x \cos^3 x \, dx$

(c) $\int \sin 4x \cos 3x \, dx$ (d) $\int \cos 4x \cos 2x \, dx$

26.5.2 Integration of Function of The Type $\frac{f'(x)}{f(x)}$

To evaluate $\int \frac{f'(x)}{f(x)} dx$, we put f (x) = t. Then f ' (x) dx = dt.

$$\therefore \quad \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|f(x)| + C$$

Integral of a function, whose numerator is derivative of the denominator, is equal to the logarithm of the denominator.

Example 26.19 Evaluate :

(i) $\int \frac{2x}{x^2+1} dx$ (ii) $\int \frac{4x^3}{5+x^4} dx$ (iii) $\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)}$

Solution :

(i) Now 2x is the derivative of x^2+1.

$\therefore$ By applying the above result, we have

$$\int \frac{2x}{x^2+1} dx = \log|x^2+1| + C$$

(ii) $4x^3$ is the derivative of $5+x^4$.

$$\therefore \quad \int \frac{4x^3}{5+x^4} dx = \log|5+x^4| + C$$

(iii) $\frac{1}{2\sqrt{x}}$ is the derivative of $3+\sqrt{x}$

$$\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)} = \log|3+\sqrt{x}| + C$$

Example 26.20 Evaluate :

(i) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$ (ii) $\int \frac{e^{2x}-1}{e^{2x}+1} dx$

Solution :

(i) $e^x + e^{-x}$ is the derivative of $e^x - e^{-x}$

$$\therefore \qquad \int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \log\left|e^x - e^{-x}\right| + C$$

Alternatively,

For $\displaystyle \int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$,

Put $e^x - e^{-x} = t.$

Then $\left(e^x + e^{-x}\right)dx = dt$

$$\therefore \qquad \int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^x - e^{-x}\right| + C$$

(ii) $\displaystyle \int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx$

Here $e^{2x} - 1$ is not the derivative of $e^{2x} + 1$. But if we multiply the numerator and denominator by e^{-x}, the given function will reduce to

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \log\left|e^x + e^{-x}\right| + C$$

$$\therefore \qquad \int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} = \log\left|e^x + e^{-x}\right| + C$$

$$\left[\because \left(e^x - e^{-x}\right) \text{ is the derivative of } \left(e^x + e^{-x}\right)\right]$$

CHECK YOUR PROGRESS 26.4

1. Evaluate :

(a) $\displaystyle \int \frac{x}{3x^2 - 2}\,dx$ (b) $\displaystyle \int \frac{2x + 1}{x^2 + x + 1}\,dx$ (c) $\displaystyle \int \frac{2x + 9}{x^2 + 9x + 30}\,dx$

(d) $\displaystyle \int \frac{x^2 + 1}{x^3 + 3x + 3}\,dx$ (e) $\displaystyle \int \frac{2x + 1}{x^2 + x - 5}\,dx$ (f) $\displaystyle \int \frac{dx}{\sqrt{x}\left(5 + \sqrt{x}\right)}$

(g) $\displaystyle \int \frac{dx}{x\left(8 + \log x\right)}$

2. Evaluate :

(a) $\int \frac{e^x}{2 + be^x} dx$ (b) $\int \frac{dx}{e^x - e^{-x}}$

26.5.3 Integration by Substitution

Example 26.21

(i) $\int \tan x dx$ (ii) $\int \sec x dx$ (iii) $\int \frac{1 - \tan x}{1 + \tan x} dx$

(iv) $\int \frac{(1 - \sin x)}{(1 + \cos x)} dx$ (v) $\int \text{cosec}^5 x \cot x dx$ (vi) $\int \frac{\sin x}{\sin(x - a)} dx$

Solution :

(i) $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$

$= -\int \frac{-\sin x}{\cos x} dx$

$= -\log|\cos x| + C$ ($\because -\sin x$ is derivative of $\cos x$)

$= \log\left|\frac{1}{\cos x}\right| + C$ or $= \log|\sec x| + C$

$\therefore \int \tan x dx = \log|\sec x| + C$

Alternatively,

$\int \tan x dx = \int \frac{\sin x \, dx}{\cos x}$

$= -\int \frac{-\sin x \, dx}{\cos x}$

Put $\cos x = t$.

Then $-\sin x \, dx = dt$

$\therefore \int \tan x dx = -\int \frac{dt}{t}$

$= -\log|t| + C$

$= -\log|\cos x| + C$

$= \log\left|\frac{1}{\cos x}\right| + C$

$= \log|\sec x| + C$

(ii) $\int \sec x \, dx$

sec x can not be integrated as such because sec x by itself is not derivative of any function. But this is not the case with $\sec^2 x$ and sec x tan x. Now $\int \sec x \, dx$ can be written as

$$\int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$$

Put $\quad \sec x + \tan x = t.$

Then $\quad (\sec x \tan x + \sec^2 x) dx = dt$

$\therefore \quad \int \sec x \, dx = \int \frac{dt}{t}$

$$= \log|t| + C$$

$$= \log|\sec x + \tan x| + C$$

(iii) $\quad \int \frac{1 - \tan x}{1 + \tan x} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Put $\quad \cos x + \sin x = t$

So that $(-\sin x + \cos x) dx = dt$

$\therefore \quad \int \frac{1 - \tan x}{1 + \tan x} dx = \int \frac{1}{t} dt$

$$= \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

(iv) $\quad \int \frac{1 + \sin x}{1 + \cos x} dt = \int \frac{1}{1 + \cos x} dx + \int \frac{\sin x}{1 + \cos x} dx$

$$= \int \frac{1}{2\cos^2\left(\frac{x}{2}\right)} dx + \int \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}$$

$$= \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx + \int \tan\frac{x}{2} dx$$

Put $\quad \frac{x}{2} = t \Rightarrow \frac{1}{2} dx = dt$

$$\int \frac{1 + \sin x}{1 + \cos x} dx = \int \sec^2 t \, dt + 2\int \tan t \, dt$$

$$= \tan t - 2\log|\cos t| + C$$

$$= \tan\frac{x}{2} - 2\log\left|\cos\left(\frac{x}{2}\right)\right| + C$$

(v) $\int \operatorname{cosec}^5 x \cot x \, dx = -\int -\operatorname{cosec}^4 x \operatorname{cosec} x \cot x \, dx$

Put $\operatorname{cosec} x = t$.

Then $-\operatorname{cosec} x \cot x \, dx = dt$

$\therefore$ $\int \operatorname{cosec}^5 x \cot x \, dx = -\int t^4 dt$

$$= -\frac{t^5}{5} + C$$

$$= \frac{-(\operatorname{cosec} x)^5}{5} + C$$

(vi) $\int \frac{\sin x}{\sin(x-a)} dx$

Put $x - a = t$

Then $dx = dt$ and $x = t + a$

$\therefore$ $\int \frac{\sin x}{\sin(x-a)} dx = \int \frac{\sin(t+a)}{\sin t} dt$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t} dt$$

$(\because \sin(A+B) = \sin A \cos B + \cos A \sin B)$

$$= \cos a \int dt + \sin a \int \cot t \, dt$$

(cos a and sin a are constants.)

$$= \cos a \cdot t + \sin a \log|\sin t| + C$$

$$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C$$

Example 26.22 Evaluate $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Solution : Put $x = a\sin\theta \Rightarrow dx = a\cos\theta \, d\theta$

$\therefore$ $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{a\cos\theta}{\sqrt{a^2 - a^2\sin^2\theta}} d\theta$

$$= \frac{1}{a}\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$$

$$= \frac{1}{a}\int \frac{1}{\cos\theta} d\theta$$

$$= \frac{1}{a}\int \sec\theta \, d\theta$$

$$= \frac{1}{a}\log\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1+\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{a+x}{\sqrt{a^2-x^2}}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{\sqrt{a+x}}{\sqrt{a-x}}\right| + C$$

$$= \frac{1}{a}\log\left|\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\right| + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Example 26.23 Evaluate : $\int \frac{1}{x^2-a^2}dx$

Solution : Put $x = a\sec\theta \quad \Rightarrow \quad dx = a\sec\theta\tan\theta \, d\theta$

$$\therefore \quad \int \frac{1}{x^2-a^2}dx = \int \frac{a\sec\theta\tan\theta \, d\theta}{a^2\sec^2\theta - a^2}$$

$$= \frac{1}{a}\int \frac{\sec\theta\tan\theta}{\tan^2\theta}d\theta \qquad \left(\tan^2\theta = \sec^2\theta - 1\right)$$

$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}d\theta$$

$$= \frac{1}{a}\int \frac{1}{\sin\theta}d\theta = \frac{1}{a}\int \operatorname{cosec}\theta \, d\theta$$

$$= \frac{1}{a}\log\left|\operatorname{cosec}\theta - \cot\theta\right| + C$$

$$= \frac{1}{a} \log \left| \frac{1 - \cos\theta}{\sin\theta} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{1 - \frac{a}{x}}{\sqrt{1 - \frac{a^2}{x^2}}} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{x - a}{\sqrt{x^2 - a^2}} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{\sqrt{x - a}}{\sqrt{x + a}} \right| + C$$

$$= \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C$$

Example 26.24 $\int \frac{1}{a^2 + x^2} dx$

Solution : Put $x = a \tan\theta \Rightarrow dx = a \sec^2\theta d\theta$

$$\therefore \quad \int \frac{1}{a^2 + x^2} dx = \int \frac{a \sec^2\theta}{a^2 (1 + \tan^2\theta)} d\epsilon$$

$$= \frac{1}{a} \int d\epsilon$$

$$= \frac{1}{a} \theta + C \qquad \left(\frac{x}{a} = \tan\epsilon \Rightarrow \tan^{-1}\frac{x}{a} = \theta \right)$$

$$= \frac{1}{a} \tan^{-1}\frac{x}{a} + C$$

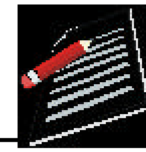
Example 26.25 $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Put $x = a \sin\theta \Rightarrow dx = a \cos\theta d\epsilon$

$$\therefore \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{a\cos\theta}{\sqrt{a^2 - a^2 \sin^2\theta}} d\epsilon$$

$$= \int \frac{a\cos\theta}{a\cos\theta} d\theta$$

$$= \int d\epsilon$$



Notes

$$= \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C$$

Example 26.26 $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx$

Solution : Let $\quad x = a\sec\theta \Rightarrow \quad dx = a\sec\theta\tan\theta\,d\theta$

$$\therefore \quad \int \frac{1}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sqrt{\sec^2\theta - 1}}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \log\left|\sec\theta + \tan\theta\right| + C$$

$$= \log\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2}\right| + C$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Example 26.27 $\int \frac{1}{\sqrt{a^2 + x^2}}\,dx$

Solution : Put $x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta\,d\theta$

$$= \int \sec\theta\,d\theta \qquad \text{[As example 26.25]}$$

$$= \log\left|\sec\theta + \tan\theta\right| + C$$

$$= \log\left|\frac{1}{a}\sqrt{a^2 + x^2} + \frac{x}{a}\right| + C$$

$$= \log\left|\sqrt{a^2 + x^2} + x\right| + C$$

Example 26.28 $\int \frac{x^2 + 1}{x^4 + 1}\,dx$

Solution : Since x^2 is not the derivative of $x^4 + 1$, therefore, we write the given integral as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\,dx$$

Let $\quad x - \frac{1}{x} = t.$

$$\therefore \quad \left(1 + \frac{1}{x^2}\right)dx = dt$$

Also $x^2 - 2 + \frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2 + \frac{1}{x^2} = t^2 + 2$

$$\therefore \quad \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \int \frac{dt}{t^2 + 2}$$

$$= \int \frac{dt}{(t)^2 + \left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) + C$$

Example 26.29 $\int \frac{x^2 - 1}{x^4 + 1} dx$

Solution : $\int \frac{x^2 - 1}{x^4 + 1} dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx$

Put $x + \frac{1}{x} = t$.

Then $\left(1 - \frac{1}{x^2}\right) dx = dt$

Also $x^2 + 2 + \frac{1}{x^2} = t^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} = t^2 - 2$

$$\therefore \quad \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \int \frac{dt}{t^2 - 2}$$

$$= \int \frac{dt}{(t)^2 - \left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| + C$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| + C$$

Example 26.30 $\int \frac{x^2}{x^4 + 1} dx$

Solution : In order to solve it, we will reduce the given integral to the integrals given in Examples 26.28 and 26.29.

i.e., $$\int \frac{x^2}{x^4 + 1} dx = \frac{1}{2} \int \left[\frac{x^2 + 1}{x^4 + 1} + \frac{x^2 - 1}{x^4 + 1} \right] dx$$

$$= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + 1} dx + \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 1} dx$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| \right] + C$$

Example 26.31 $\int \frac{1}{x^4 + 1} dx$

Solution : We can reduce the given integral to the following form

$$\frac{1}{2} \int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 + 1} dx$$

$$= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + 1} dx - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 1} dx$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| \right] + C$$

Example 26.32 (a) $\int \frac{1}{x^2 - x + 1} dx$ (b) $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

Solution : (a) $$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{x^2 - x + \frac{1}{4} - \frac{1}{4} + 1} dx$$

$$= \int \frac{1}{\left(x - \frac{1}{2} \right)^2 + \frac{3}{4}} dx$$

$$= \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

(b) $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$

Put $x + \frac{1}{x} = t. \quad \Rightarrow \quad \left(1 - \frac{1}{x^2}\right) dx = dt$

Also $x^2 + 2 + \frac{1}{x^2} = t^2$

$\Rightarrow \quad x^2 + 1 + \frac{1}{x^2} = t^2 - 1$

$\therefore \quad \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx = \int \frac{dt}{t^2 - 1}$

$$= \frac{1}{2} \log \left|\frac{t - 1}{t + 1}\right| + C$$

$$= \frac{1}{2} \log \left|\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right| + C$$

Example 26.33 $\int \sqrt{\tan x}\, dx$

Solution : Let $\tan x = t^2 \quad \Rightarrow \quad \sec^2 x\, dx = 2t\, dt$

$\Rightarrow \quad dx = \frac{2t}{\sec^2 x} dt$

$$= \frac{2t}{1 + t^4} dt$$

$$\therefore \qquad \int \sqrt{\tan x}\, dx = \int t\left(\frac{2t}{1+t^4}\right) dt$$

$$= \int \frac{2t^2}{1+t^4} dt$$

$$= \int \left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1}\right) dt$$

$$= \int \frac{t^2+1}{t^4+1} dt + \int \frac{t^2-1}{t^4+1} dt$$

Proceed according to example 26.28 and Example 26.29 solved before.

Example 26.34 $\int \sqrt{\cot x}\, dx$

Solution : Let $\cot x = t^2 \;\Rightarrow\; -\text{cosec}^2 x\; dx = 2t\; dt$

$$\Rightarrow \qquad dx = \frac{-2t}{\text{cosec}^2 x} dt$$

$$= -\frac{2t}{t^4+1} dt$$

$$\therefore \qquad \int \sqrt{\cot x}\; dx = -\int t\left(\frac{2t}{t^4+1}\right) dt$$

$$= -\int \frac{2t^2}{t^4+1} dt$$

$$= -\int \left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1}\right) dt$$

Proceed according to Examples 26.28 and 26.29 solved before.

Example 26.35 $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

Let $\quad \sin x - \cos x = t \;\Rightarrow\; (\cos x + \sin x)\, dx = dt$

Also $\quad 1 - 2\sin x\cos x = t^2$

$$\Rightarrow \qquad 1 - t^2 = 2\sin x \cos x$$

$$\Rightarrow \qquad \frac{1-t^2}{2} = \sin x \cos x$$

$$\therefore \int \frac{\sin x - \cos x}{\sqrt{\cos x \sin x}}\,dx = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}}$$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\,\sin^{-1}[\sin x - \cos x] + C$$

(Using the result of Example 26.25)

Example 26.36 Evaluate :

(a) $\displaystyle\int \frac{dx}{\sqrt{8+3x-x^2}}$ (b) $\displaystyle\int \frac{dx}{x(1-2x)}$

Solution :

(a) $$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-\left(x^2-3x\right)}}$$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$$= \sin^{-1}\left[\frac{\left(x-\frac{3}{2}\right)}{\frac{\sqrt{41}}{2}}\right] + C$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + C$$

(b) $$\int \frac{dx}{x(1-2x)} = \int \frac{dx}{\sqrt{x-2x^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{x}{2}-x^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{1}{16}-\left[x^2-\frac{x}{2}+\frac{1}{16}\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left\{ \frac{\left(x - \frac{1}{4}\right)}{\left(\frac{1}{4}\right)} \right\} + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} (4x - 1) + C$$

CHECK YOUR PROGRESS 26.5

1. Evaluate :

(a) $\int \frac{x^2}{x^2 - 9} dx$ (b) $\int \frac{e^x}{e^{2x} + 1} dx$ (c) $\int \frac{x}{1 + x^4} dx$

(d) $\int \frac{dx}{\sqrt{16 - 9x^2}}$ (e) $\int \frac{dx}{1 + 3\sin^2 x}$ (f) $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$

(g) $\int \frac{dx}{3x^2 + 6x + 21}$ (h) $\int \frac{dx}{\sqrt{5 - 4x - x^2}}$ (i) $\int \frac{dx}{x\sqrt{3x^2 - 12}}$

(j) $\int \frac{d\theta}{\sin^4 \theta + \cos^4 \theta}$ (k) $\int \frac{e^x dx}{\sqrt{1 + e^{2x}}}$ (l) $\int \sqrt{\frac{1 + x}{1 - x}}\, dx$

(m) $\int \frac{dx}{\sqrt{2ax - x^2}}$ (n) $\int \frac{3x^2}{\sqrt{9 - 16x^6}} dx$ (o) $\int \frac{(x + 1)}{\sqrt{x^2 + 1}} dx$

(p) $\int \frac{dx}{\sqrt{9 + 4x^2}}$ (q) $\int \frac{\sin\theta}{\sqrt{4\cos^2 \theta - 1}} d\theta$ (r) $\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 4}} dx$

(s) $\int \frac{1}{(x + 2)^2 + 1} dx$ (t) $\int \frac{1}{\sqrt{16x^2 + 25}} dx$

26.6 INTEGRATION BY PARTS

In differentiation you have learnt that

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + g\frac{d}{dx}(f)$$